Crypto 2021

A Logarithmic Lower Bound for Oblivious RAM (for all parameters)

August 20, 2021

Wei-Kai Lin

Cornell University

Ilan Komargodski

האוניברסיטה העברית בירושלים
THE HEBREW UNIVERSITY OF JERUSALEM

NTT Research

Access pattern
Leaks data

Frequency,
Correlation

ORAM, Correctness

[Goldreich-Ostrovsky ’87,96]

ORAM operations (array):

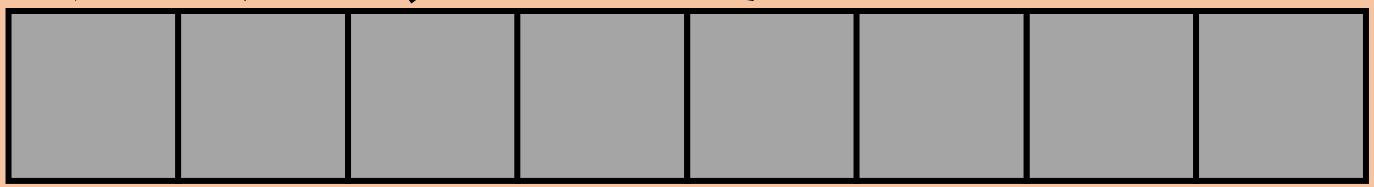
* Update(i, )
* Query(i)

“Online”:
Answer a query
before next

(implementation)

“**Probe**” memory “**cells**”

[Yao ‘81]

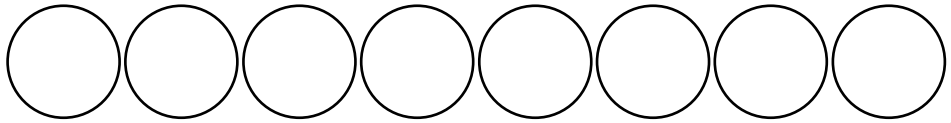


ORAM, Security

[Goldreich-Ostrovsky '87,96]

Any sequence of
Update / Query

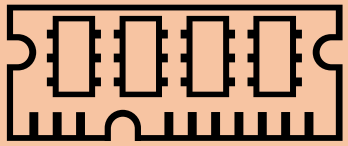
ORAM (array):

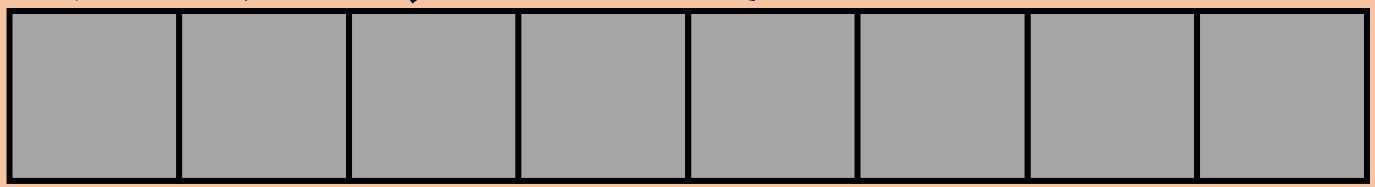


Computationally indistinguishable
Read / Write sequence

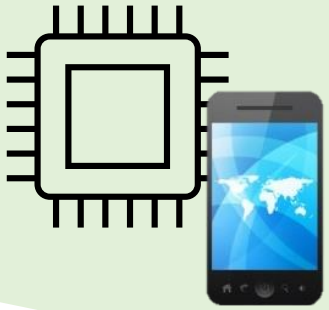
(stronger alternative: identical distribution)

Probed
locations





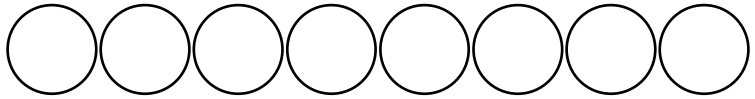
ORAM, Parameters



64-bit program:

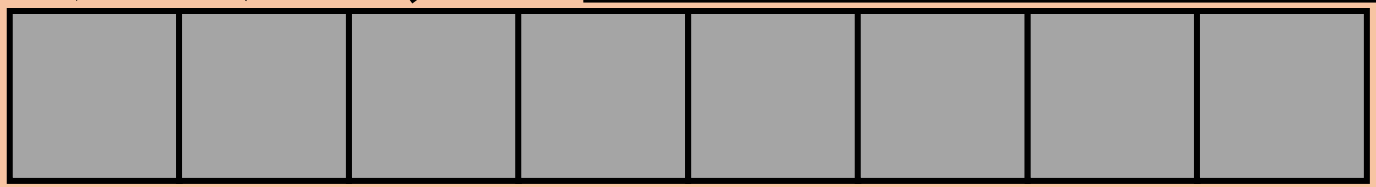
Array of n entries, each b bits

ORAM (array):

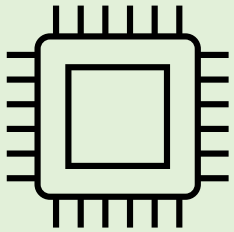


Network packets:

Each cell w bits

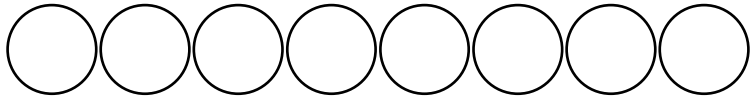


ORAM, Efficiency



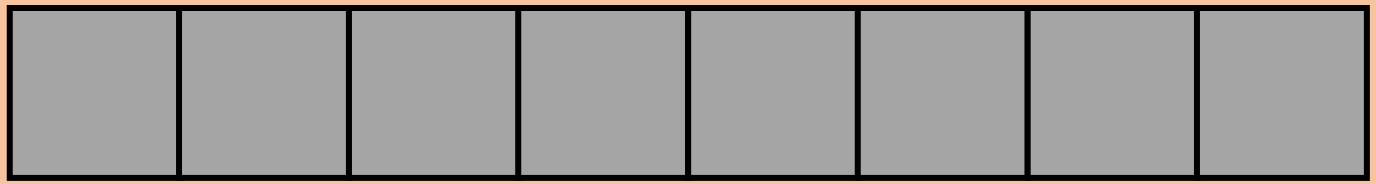
b-bit operation

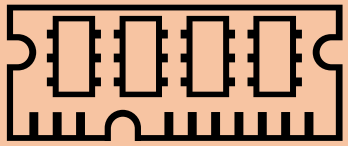
ORAM, size n:

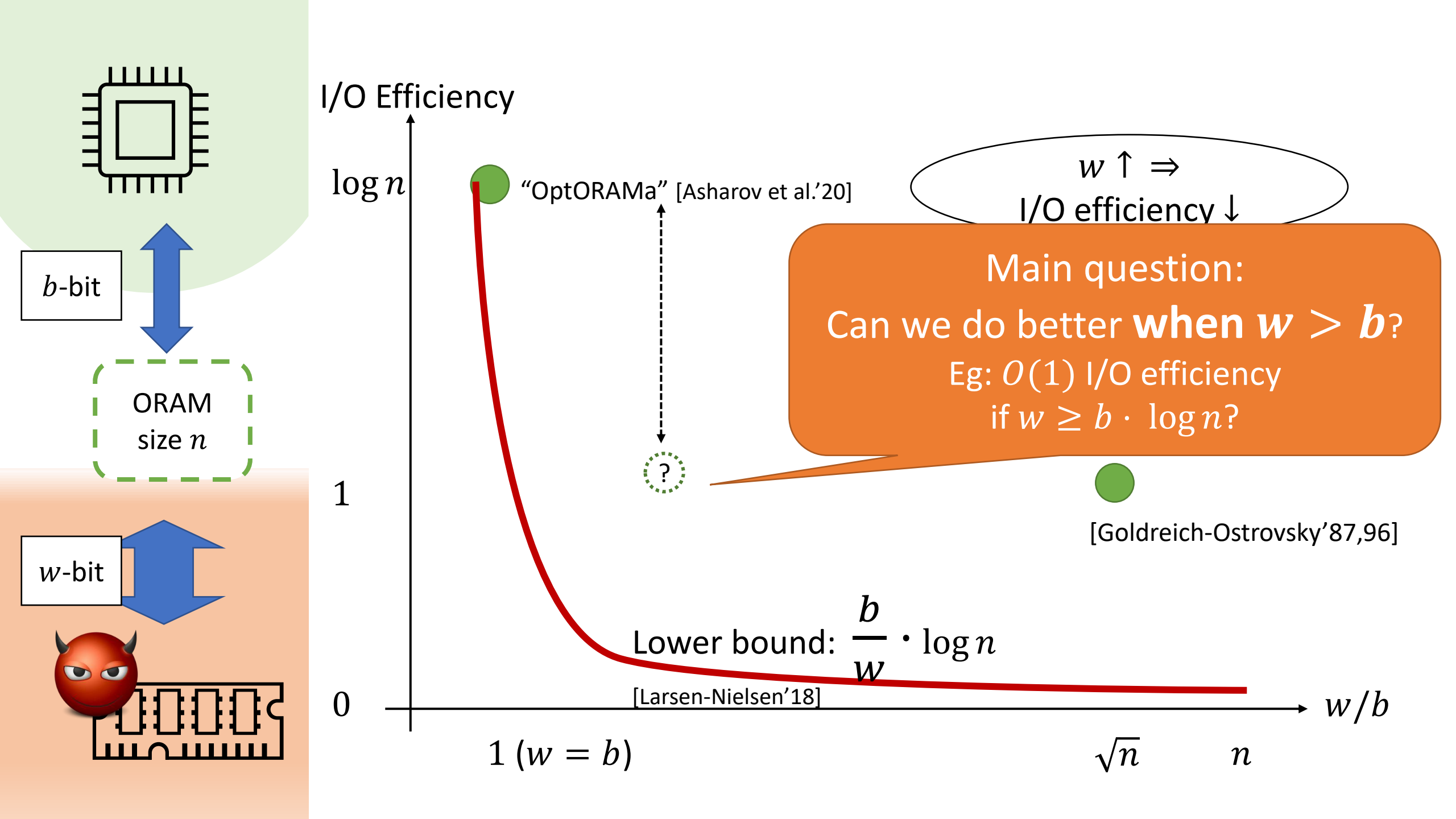


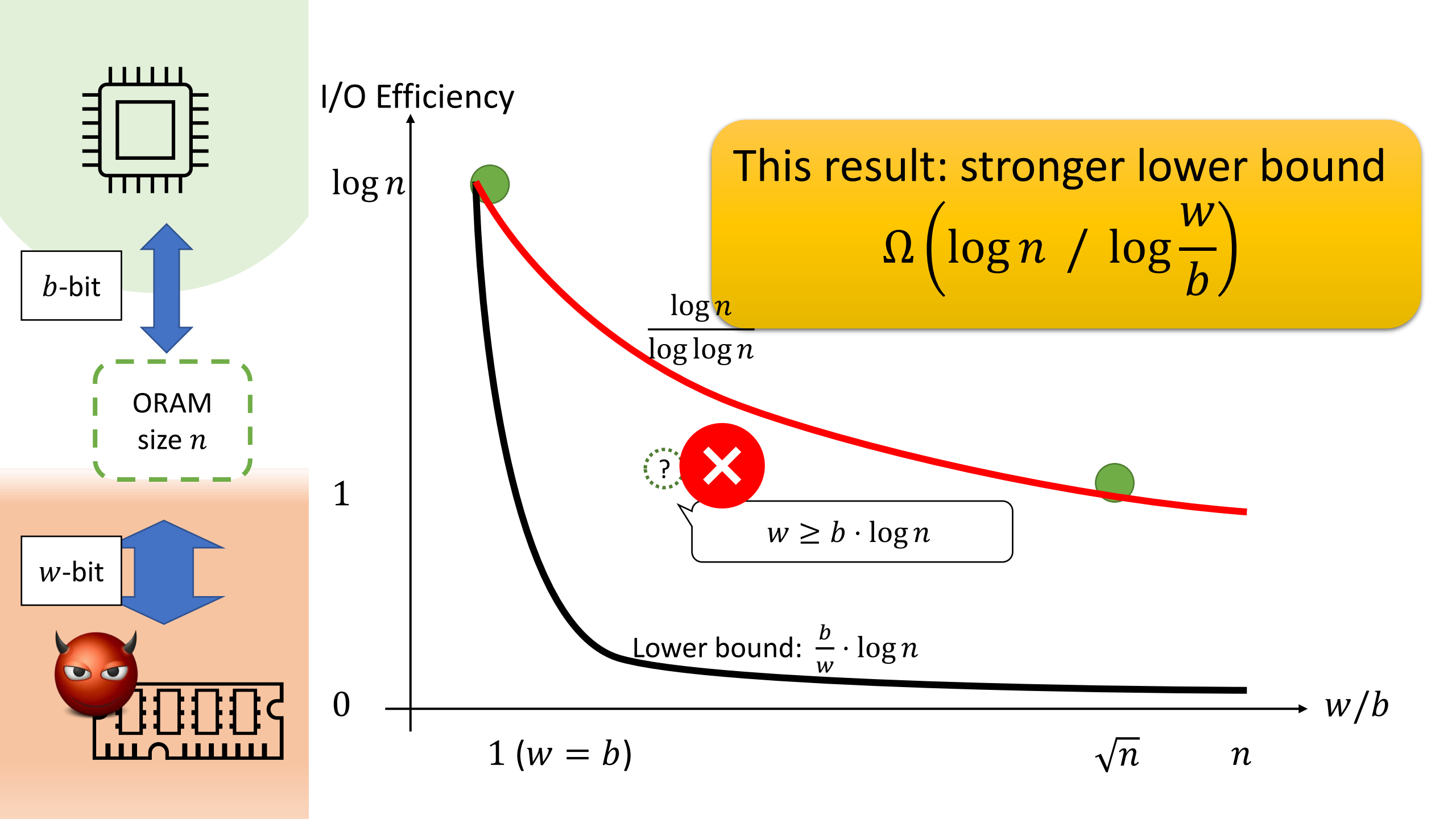
I/O Efficiency:
Num. probes per operation

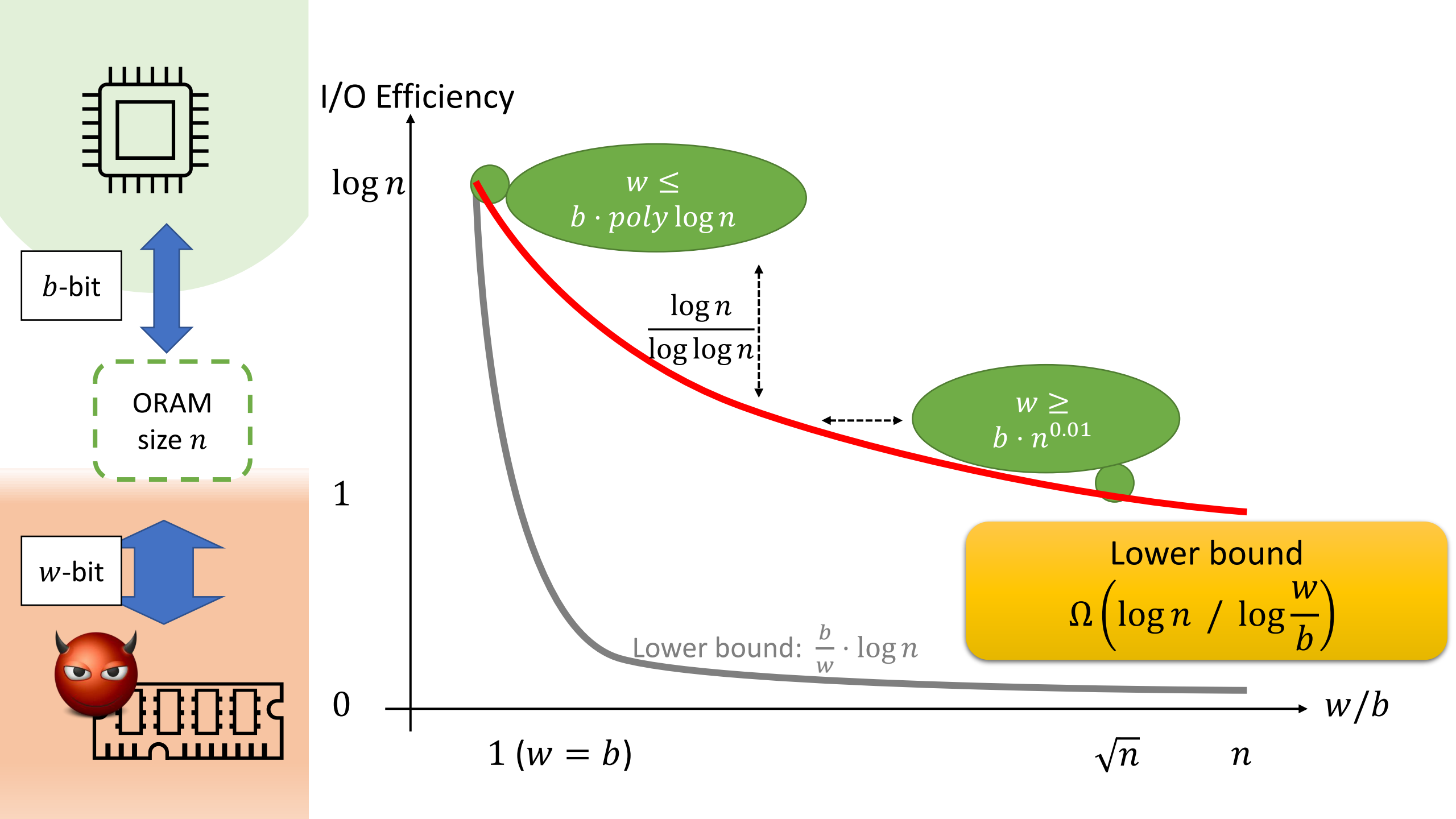
w-bit probe



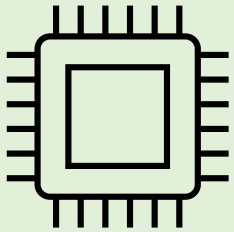



I/O Efficiency
$\log n$
1
0
"OptORAMa" [Asharov et al.'20]
?
$w \uparrow \Rightarrow$
I/O efficiency $\downarrow$
Main question:
Can we do better **when $w > b$**?
Eg: $O(1)$ I/O efficiency
if $w \geq b \cdot \log n$?
[Goldreich-Ostrovsky'87,96]
Lower bound: $\frac{b}{w} \cdot \log n$
[Larsen-Nielsen'18]
w/b
$1\ (w = b)$
$\sqrt{n}$
n
b-bit
ORAM
size n
w-bit


b-bit
ORAM
size n
w-bit
I/O Efficiency
log n
1
0
log n / log log n
?
w ≥ b · log n
Lower bound: b/w · log n
w/b
1 (w = b)
√n
n
This result: stronger lower bound
Ω(log n / log w/b)


b-bit
ORAM
size n
w-bit
I/O Efficiency
log n
1
0
w ≤
b · poly log n
log n / log log n
w ≥
b · n^0.01
Lower bound: b/w · log n
Lower bound
Ω(log n / log w/b)
w/b
1 (w = b)
√n
n

Lower Bound Proof

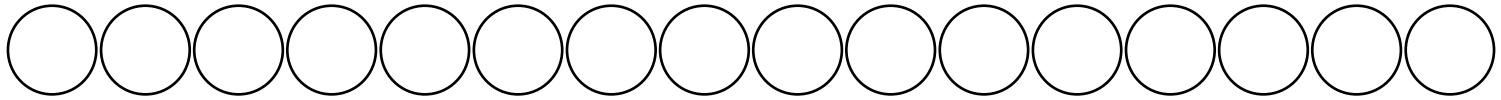


1. Update

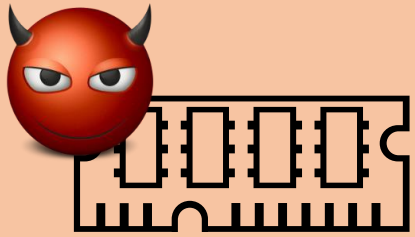
2. Query

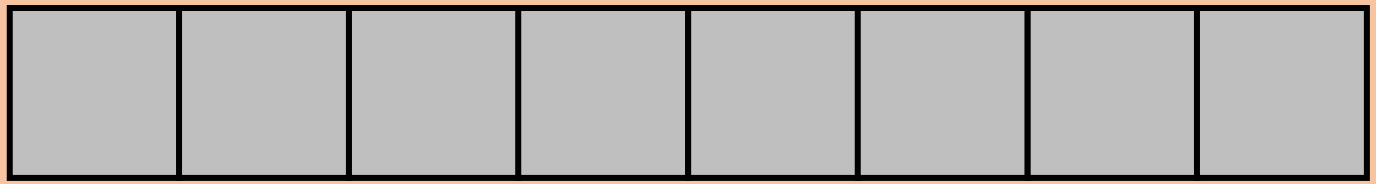
b-bit

ORAM
size n

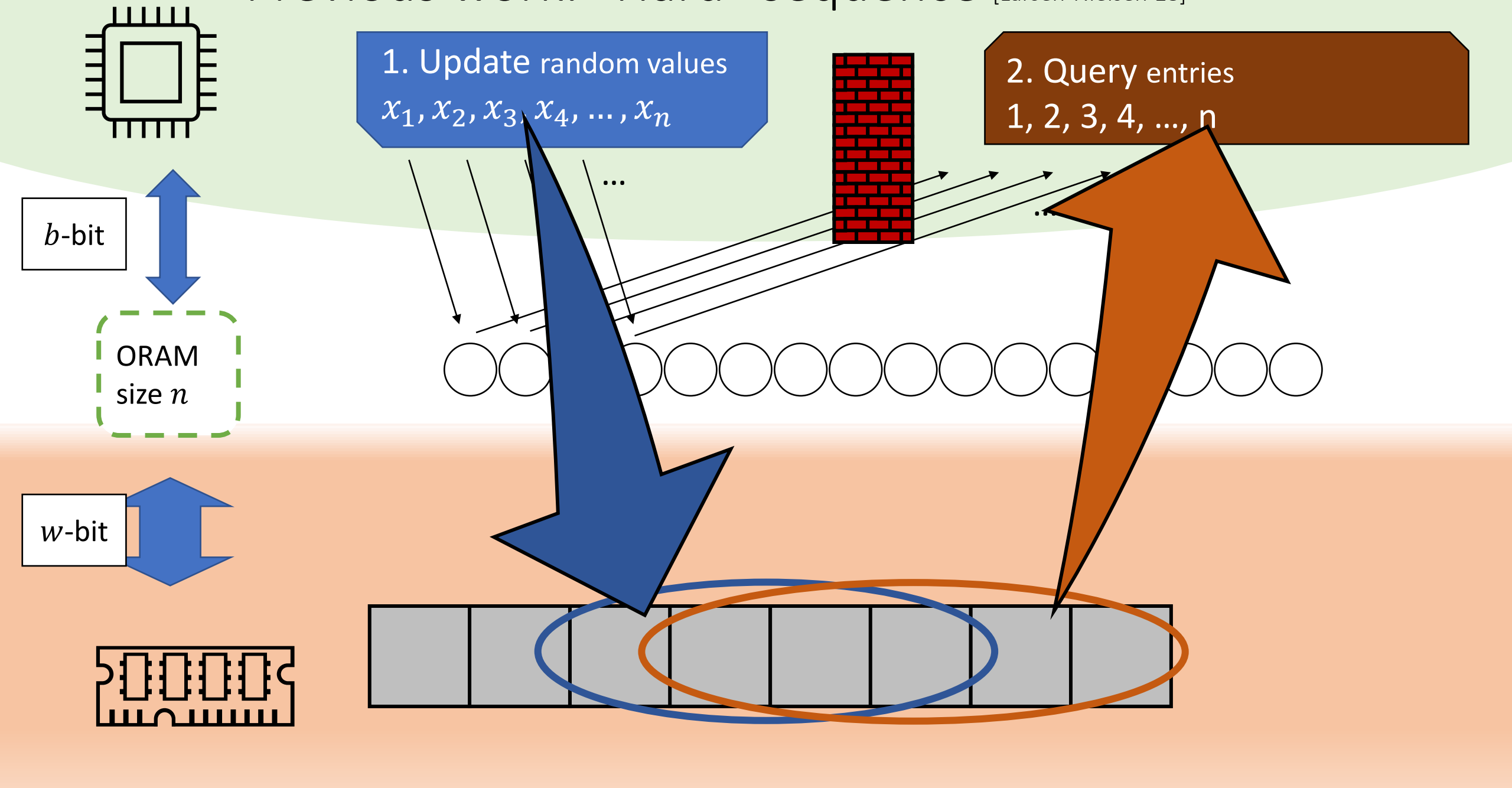


w-bit





Previous work: “Hard” sequence [Larsen-Nielsen’18]



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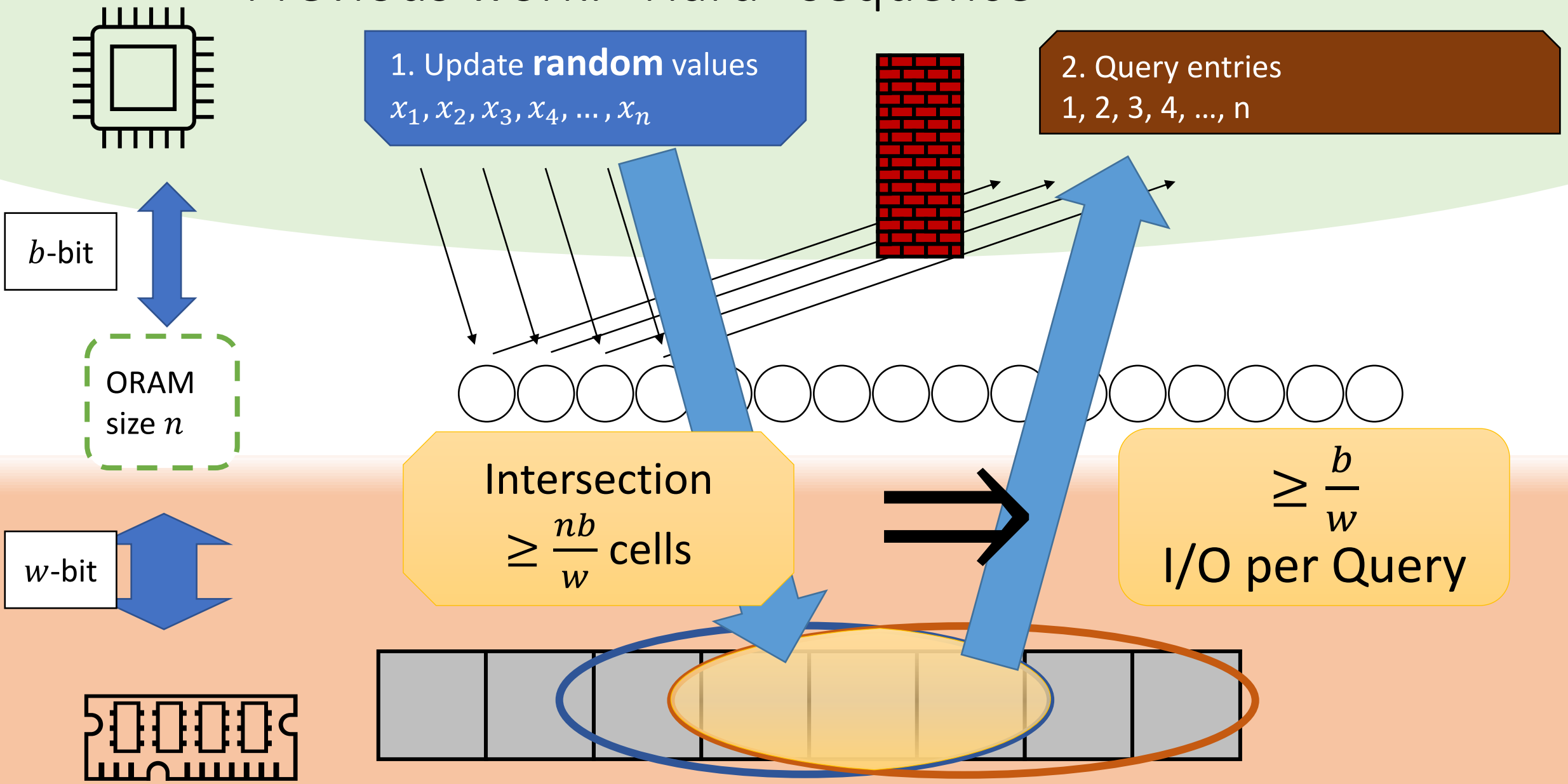
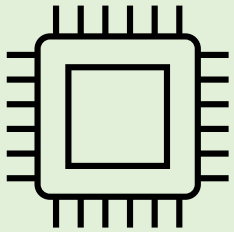
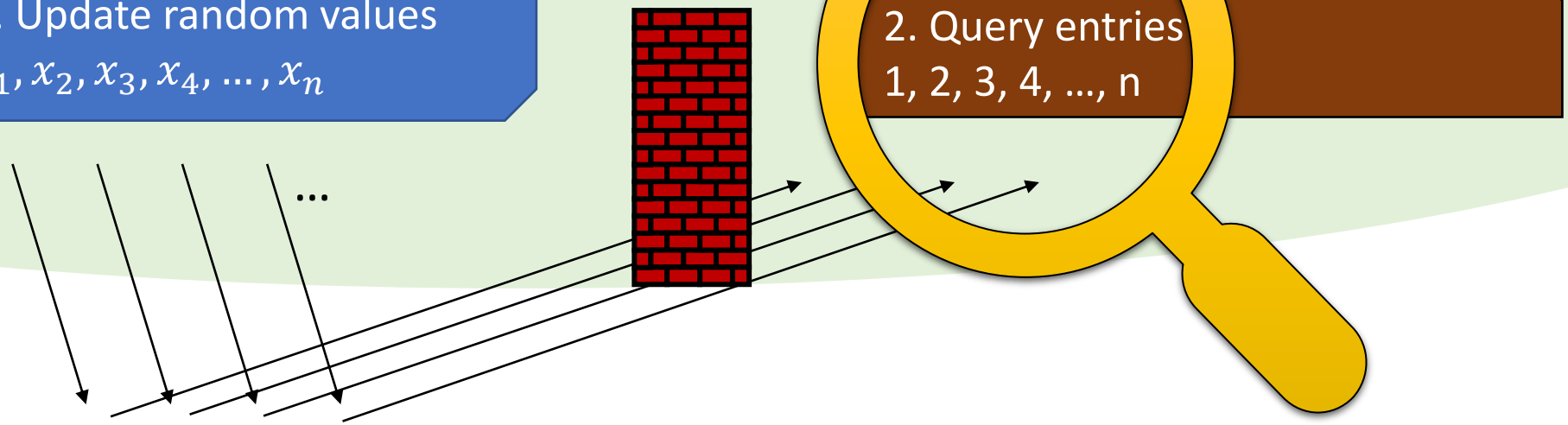


Key Idea: **Random** Queries



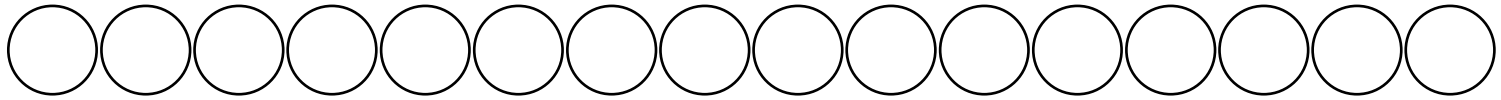
1. Update random values
$x_1, x_2, x_3, x_4, \ldots, x_n$

2. Query entries
1, 2, 3, 4, …, n



b-bit

ORAM
size n

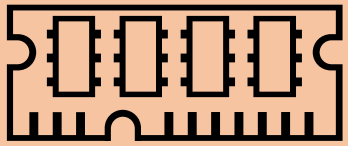


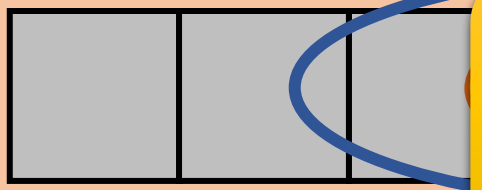
w-bit

Intersection
$\geq n \cdot \frac{b}{w}$ cells

$\Rightarrow$

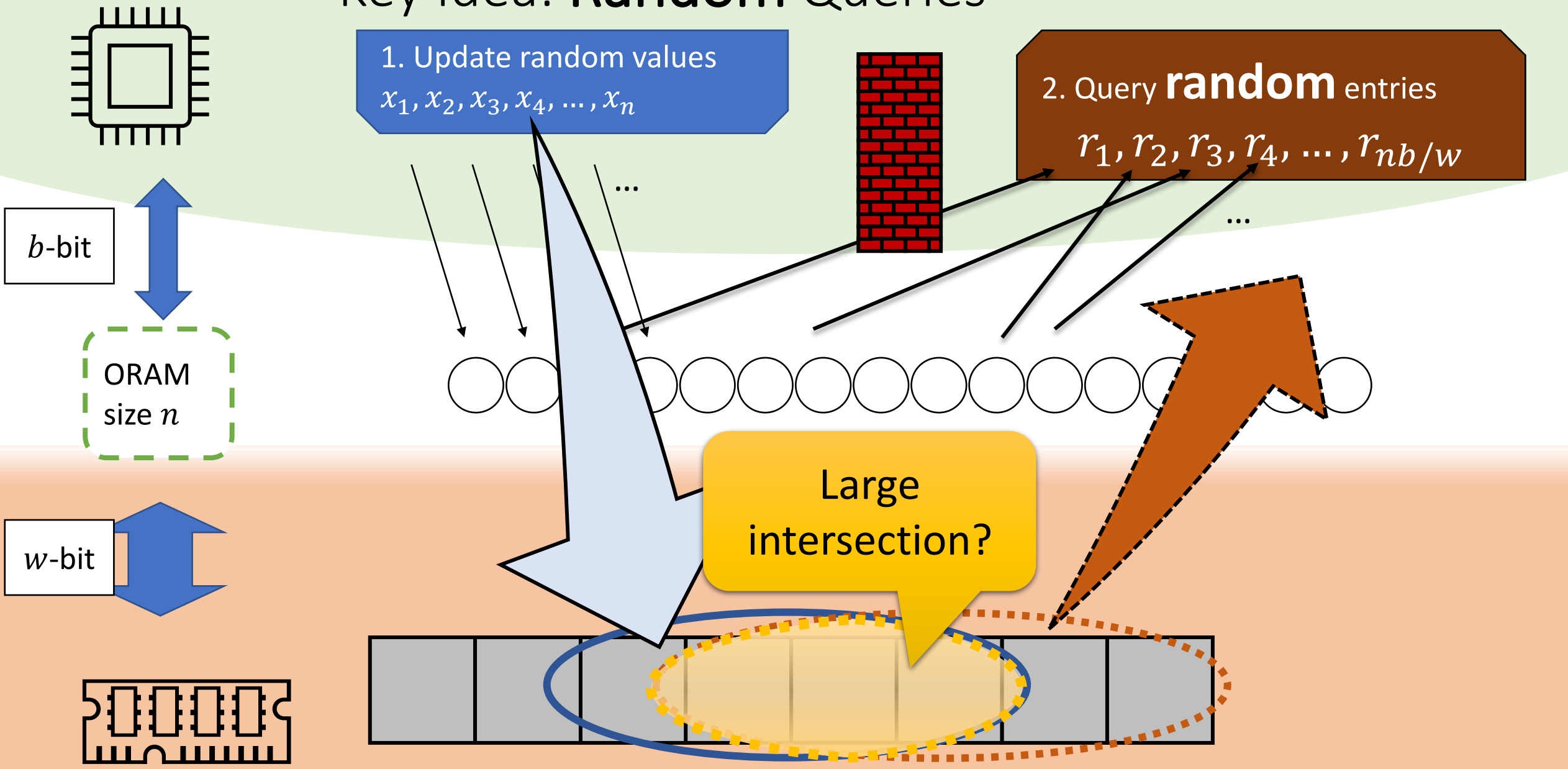
$\geq \frac{b}{w}$
I/O per query



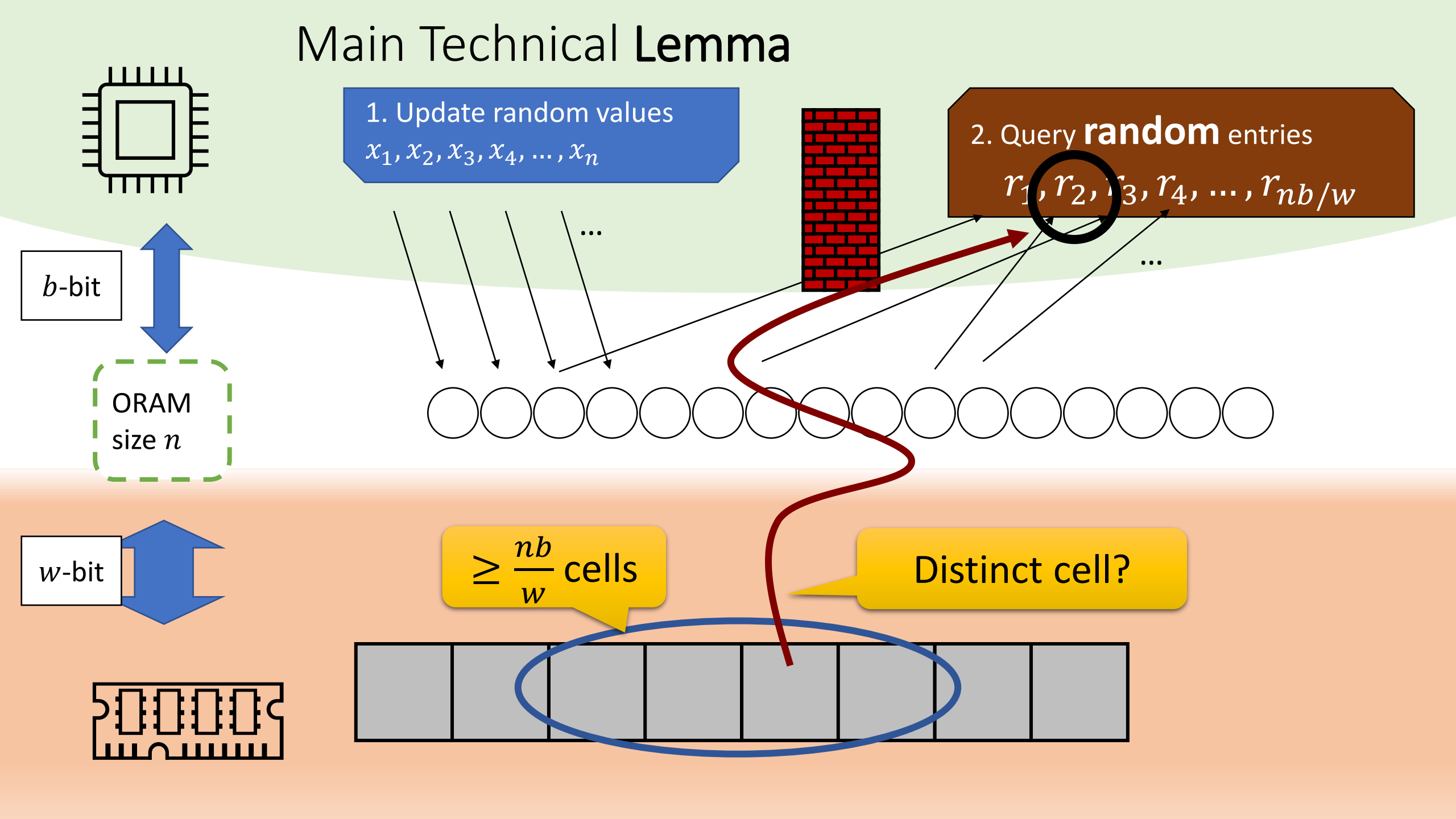


If $w \gg 100\ b$, then I/O $\ll 0.01$?
Too good to be true!

Key Idea: **Random** Queries



Main Technical **Lemma**



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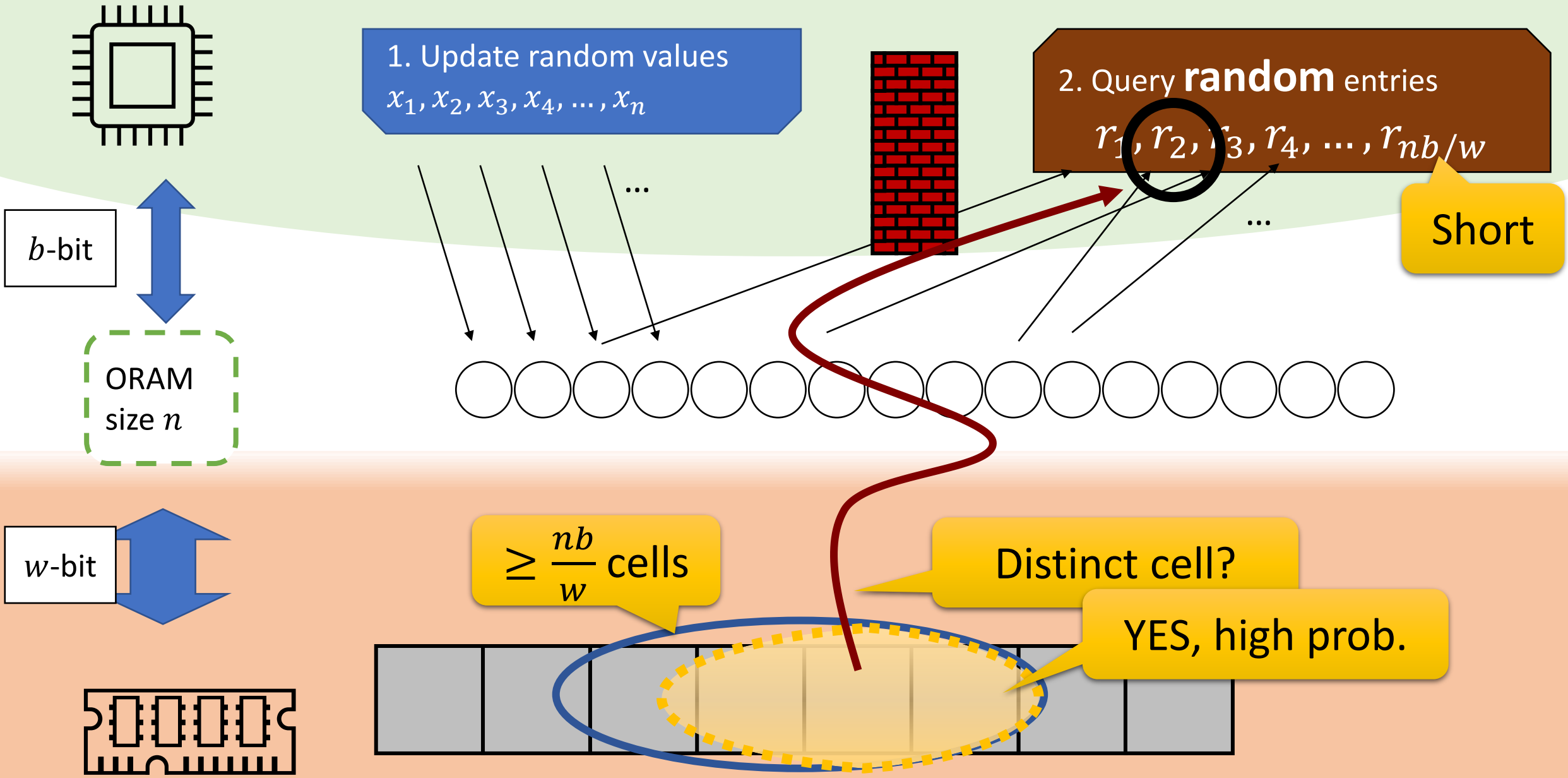

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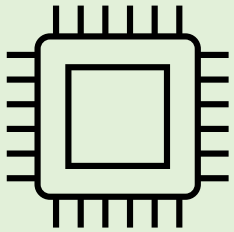

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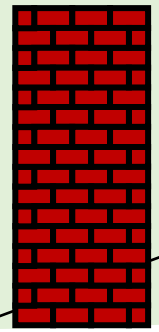


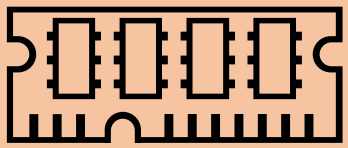
2. Query **random** entries
$r_1, r_2, r_3, r_4, \ldots, r_{nb/w}$

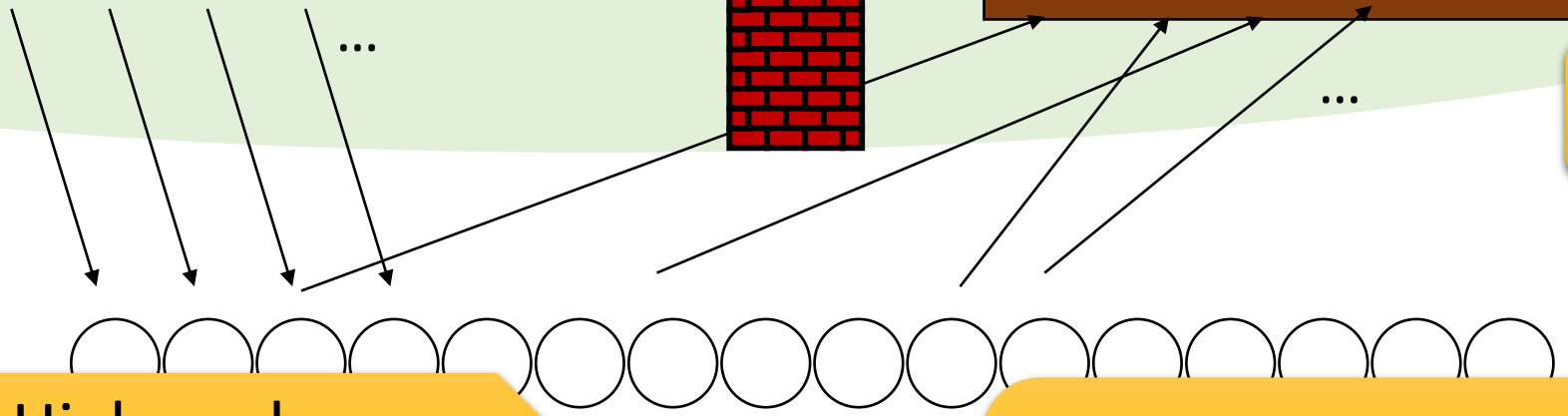
Short

b-bit

ORAM
size n

w-bit

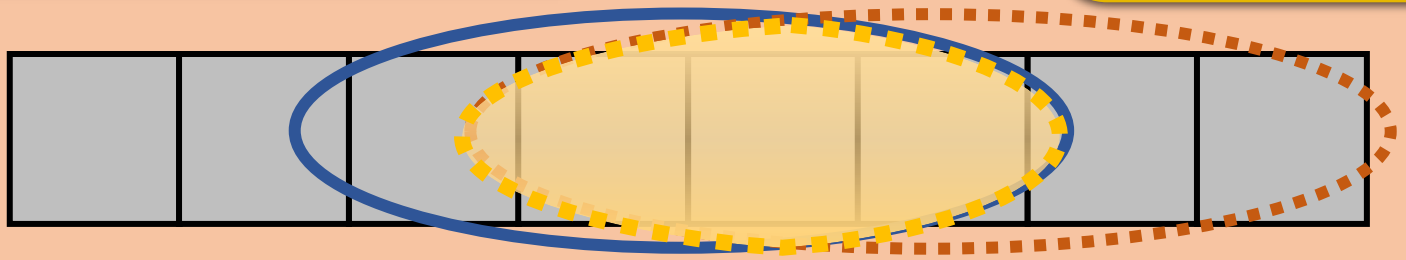




High prob:
Intersection
$= \Omega\left(\frac{nb}{w}\right)$ cells

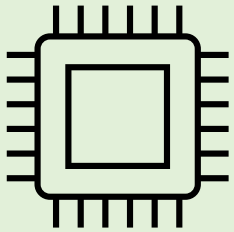
$\Rightarrow$

High prob:
$\geq \Omega(1)$ I/O per Query



Proved by compression argument

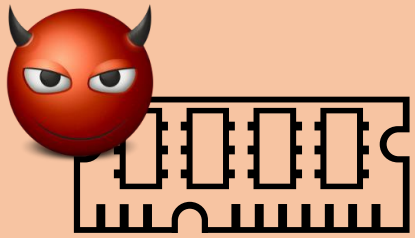
Boost by **Security**



b-bit

ORAM
size n

w-bit



Update n values

Query nb/w entr.

padding

Update n values

padding

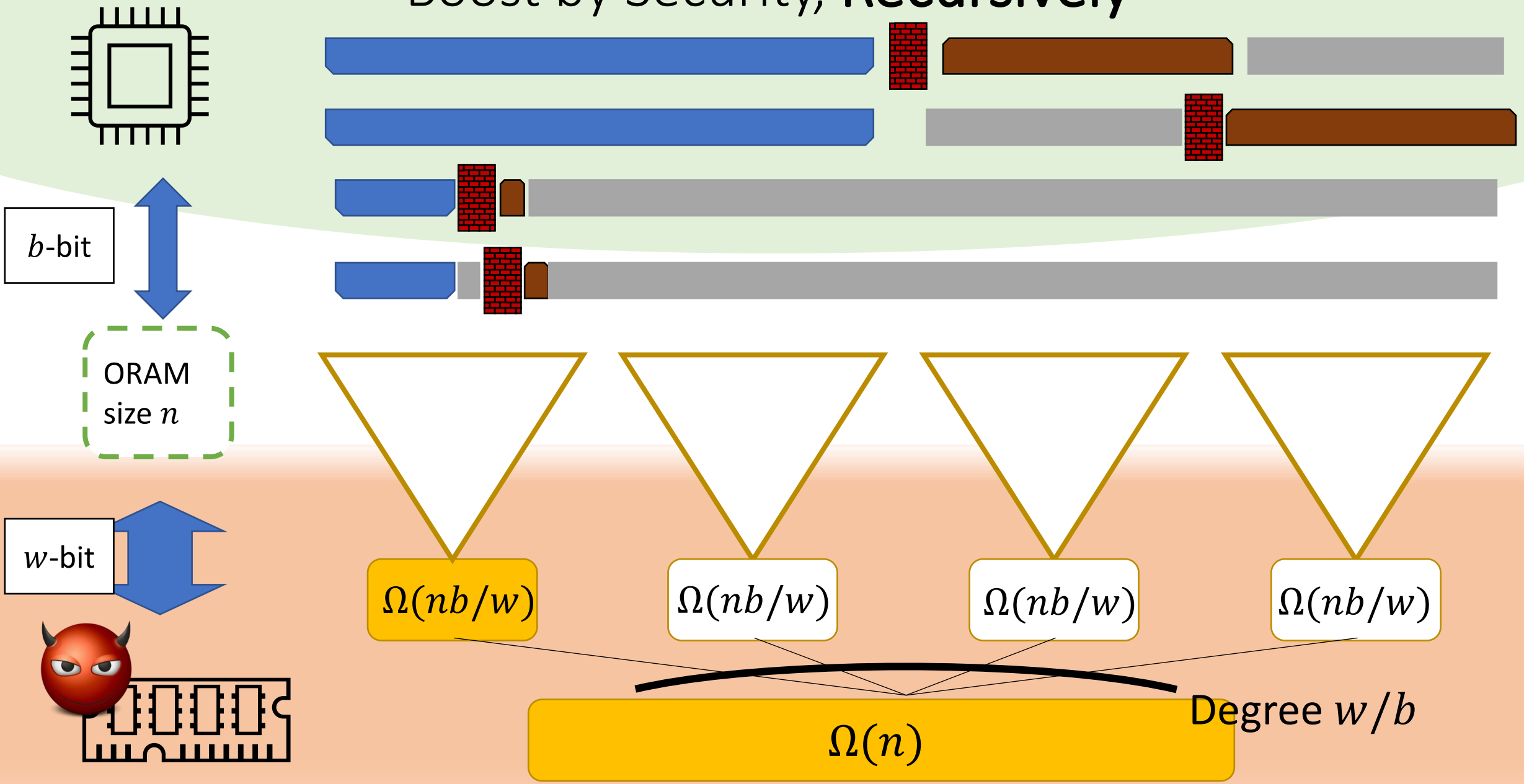
Query nb/w entr.

⋮

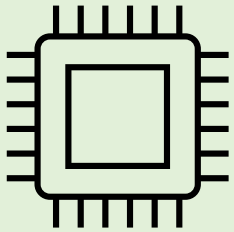
Cells: $\Omega(nb/w)$ + Cells: $\Omega(nb/w)$ + ⋯

$= \Omega(n)$

Boost by Security, **Recursively**



Boost by Security, **Recursively**



b-bit

ORAM
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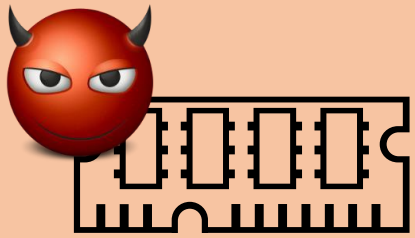


Tree height = $\log_{w/b} n$

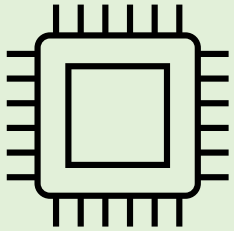
$\Rightarrow$ #I/O = $\Omega(n \cdot \log_{w/b} n)$

per n Query/Update

$\Omega(nb/w)$ $\Omega(nb/w)$ $\Omega(nb/w)$ $\Omega(nb/w)$

Degree w/b

$\Omega(n)$



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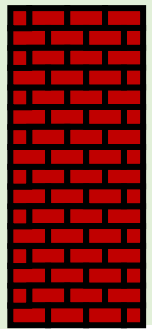

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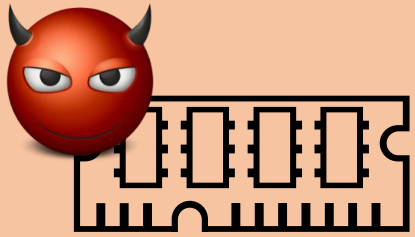
b-bit

ORAM
size n

w-bit



Main lemma (this hard sequence):

With high prob: intersection $= \Omega\left(n \cdot \frac{b}{w}\right)$ cells

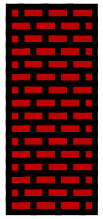
Main result (any ORAM):

Any $b \geq w$, I/O = $\Omega\left(\log n \;/\; \log \frac{b}{w}\right)$

- Unconditional (not “balls-and-bins” model)
- Computational (ORAM may use any crypto)

Challenge to main lemma

1. Update random values
$x_1, x_2, x_3, x_4, \ldots, x_n$



2. Query **random** entries
$r_1, r_2, r_3, r_4, \ldots, r_{nb/w}$

With high prob:
intersection $= \Omega\left(n \cdot \frac{b}{w}\right)$ cells

Suppose not, then exists ORAM:
Intersection $\leq 0.01\ n \cdot \frac{b}{w}$
$\Rightarrow$ Can compress random values
$x_1, x_2, x_3, x_4, \ldots, x_n$
To $< 0.99\ nb$ bits (impossible)

Alice (impossible compress) [Pătraşcu,Demaine'06]

1. If Intersection of ($x_1, \ldots, x_n$; $r_1, r_2, \ldots, r_{nb/w}$) is large, then output ($x_1, \ldots, x_n$) directly;
   Else, continue.
2. **Write** small Intersection (of cell contents, $0.01nb$ bits)
3. Pick random t from 1 to nb/w.
4. For each $\boldsymbol{i}$ from 1 to n:
   If Query($r_1, r_2, \ldots, r_{t-1}, \boldsymbol{i}$)
   can NOT be answered by small Intersection,
   then **Write** x_i

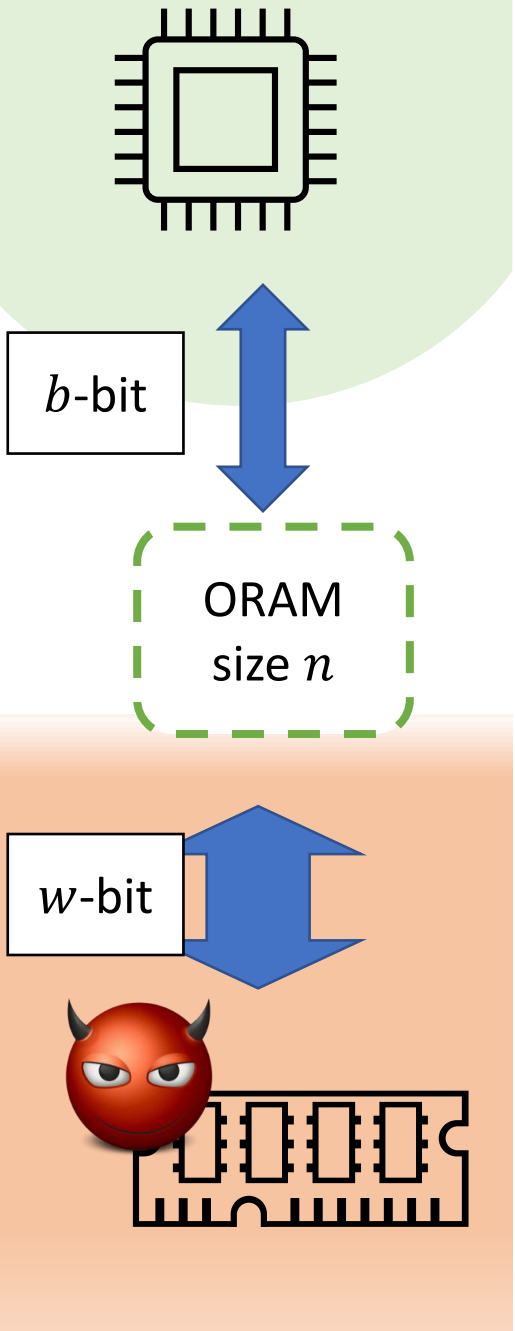
Analysis, simplified

- X, Y independent random variables
- Y^* random variable, independent and identically distributed to Y
- $f(x, y)$ arbitrary Boolean function

Then:

$$\Pr[f(X, Y^*) = 1 \mid f(X, Y) = 1] \quad \geq \quad \Pr[f(X, Y) = 1]$$

A “win” makes it more likely to “win”



Main result (any ORAM):

Any $w \geq b$, I/O = $\Omega\left(\log n \;/\; \log \frac{w}{b}\right)$

(extends to multi-server setting)

Open problems:

- Remaining gap (for computational security)
- Lower/upper bound for
  - Weaker notions (eg, differential-private ORAMs)
  - Stronger notions (eg, statistical security)

Related **new results**:

- ORAM with Worst-Case Logarithmic Overhead (Crypto2021)
- Optimal Oblivious *Parallel* RAM

Thank you!